

# THE BASE COMPONENTS OF THE DUALIZING SHEAF OF A CURVE ON A SURFACE

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**ABSTRACT.** This note studies the structure of the divisorial fixed part of  $|\omega_D|$  for a 1-connected curve  $D$  on a smooth surface  $S$ . It is shown that if the divisorial fixed part  $F$  of  $|\omega_D|$  is non empty then it has arithmetic genus  $\leq 0$  and each component of  $F$  is a smooth rational curve. The structure of curves  $D$ , with non empty divisorial fixed part  $F$  for  $|\omega_D|$ , is also described.

## INTRODUCTION

In this note we study the structure of the fixed part of  $|\omega_D|$  for a 1-connected curve  $D$  on a smooth surface  $S$ . It is well known that if  $|\omega_D|$  has base points then  $D$  is not 2-connected (cf. [CFM]), and in fact there has been work of several authors concerning the structure of  $\omega_D$  (see, e.g., [CCFR], [CFM], [K], [M]) but as far as we know the present result is new.

We prove the following theorem:

**Theorem 0.1.** *Let  $D$  be a 1-connected curve on a smooth algebraic projective surface  $S$  over  $\mathbb{C}$ . If  $0 \prec Z \prec D$  is a curve contained in the fixed part of  $|\omega_D|$  then:*

- i) *every irreducible component of  $Z$  is a smooth rational curve;*
- ii) *for any  $0 \prec Z' \preceq Z$ ,  $p_a(Z') \leq 0$  and  $h^1(Z', \mathcal{O}_{Z'}) = 0$ ;*
- iii)  *$p_a(Z) = 0$  if and only if  $Z$  is 1-connected;*
- iv)  *$h^0(D - Z, \mathcal{O}_{D-Z}) = (D - Z)Z + p_a(Z)$ .*

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Furthermore if  $p_a(Z) = 0$  and  $Z(D - Z) = m$ ,  $D - Z$  decomposes as  $B_1 + \dots + B_m$ , such that  $\mathcal{O}_{B_i}(-(B_{i+1} + \dots + B_m)) = \mathcal{O}_{B_i}$ , for every  $i = 1, \dots, m-1$ ,  $B_i Z = 1$  and  $B_i$  is 1-connected, for every  $i = 1, \dots, m$ . In addition either  $B_i \preceq B_{i+1} + \dots + B_m$  or  $B_i \cap B_{i+1} + \dots + B_m = \emptyset$ .

## 1. PRELIMINARIES

**1.1. Notation.** By a *curve* we mean a non-zero effective divisor on a smooth algebraic projective surface  $S$  over  $\mathbb{C}$ .  $K$  will denote the canonical bundle of  $S$ .

Given a curve  $D$ ,  $\omega_D$  denotes its dualizing sheaf and  $p_a(D)$  its arithmetic genus.

A curve  $D$  is  $m$ -connected if for every decomposition  $D = A + B$ , with  $A, B$  effective non-zero divisors,  $AB \geq m$ .

For any invertible sheaf  $\mathcal{L}$  on  $D$  we denote by  $h^i(D, \mathcal{L})$  the dimension as a  $\mathbb{C}$ -vector space of  $H^i(D, \mathcal{L})$ .

**1.2. Some properties.** Here we list some properties that will be used throughout without further reference.

- Given a curve  $D$ ,  $2p_a(D) - 2 = KD + D^2$  (adjunction formula).
- $h^0(D, \mathcal{O}_D) - h^1(D, \mathcal{O}_D) = 1 - p_a(D)$ .
- By duality one has  $h^0(D, \mathcal{O}_D) = h^1(D, \omega_D)$  and  $h^1(D, \mathcal{O}_D) = h^0(D, \omega_D)$ .
- If a curve  $D$  decomposes as the sum of two curves  $D_1, D_2$  then  $2p_a(D) - 2 = KD + D^2 = KD_1 + KD_2 + D_1^2 + D_2^2 + 2D_1D_2 = 2p_a(D_1) - 2 + 2p_a(D_2) - 2 + 2D_1D_2$  and so  $p_a(D) = p_a(D_1) + p_a(D_2) - 1 + D_1D_2$ .
- ([CFM]) Let  $D$  be an  $m$ -connected curve on the surface  $S$  and let  $D = D_1 + D_2$  with  $D_1, D_2$  curves. Then:
  - i) if  $D_1 \cdot D_2 = m$ , then  $D_1$  and  $D_2$  are  $[(m+1)/2]$ -connected;
  - ii) if  $D_1$  is chosen to be minimal subject to the condition  $D_1 \cdot (D - D_1) = m$ , then  $D_1$  is  $[(m+3)/2]$ -connected.

## 1.3. Auxiliary results.

**Definition 1.1.** Let  $D$  be a reducible curve on a smooth surface  $S$ ,  $\mathcal{L}$  an invertible sheaf on  $D$  and let  $s \in H^0(D, \mathcal{L})$  with  $s \neq 0$  such that  $s$  vanishes identically on some component of  $D$ . Let  $Z_s \prec D$  be the biggest curve such that  $s|_{Z_s} \equiv 0$ .

We will say that  $s$  is 0-maximal if there is no global section  $t$  of  $\mathcal{L}$  with  $Z_s \prec Z_t$ .

**Lemma 1.2.** *Let  $A$  be a curve such that  $h^0(A, \mathcal{O}_A) \geq 2$ . Then there is a decomposition  $A = A_1 + A_2$  where  $A_1, A_2$  are curves such that*

- i)  $h^0(A_1, \mathcal{O}_{A_1}(-A_2)) \neq 0$ ;
- ii)  $A_1 A_2 \leq 0$ ;
- iii) for each component  $\Gamma$  of  $A_1$ ,  $\Gamma A_2 \leq 0$ ;
- iv) for each component  $\Gamma$  of  $A_1$ , the restriction map

$$H^0(A_1, \mathcal{O}_{A_1}(-A_2)) \rightarrow H^0(\Gamma, \mathcal{O}_\Gamma(-A_2))$$

*is injective;*

- v) for each component  $\Gamma$  of  $A_1$ ,

$$h^0(A, \mathcal{O}_A) \leq h^0(A_2, \mathcal{O}_{A_2}) + h^0(\Gamma, \mathcal{O}_\Gamma(-A_2)).$$

Furthermore if  $A_1 A_2 = 0$  then  $\mathcal{O}_{A_1}(-A_2) = \mathcal{O}_{A_1}$ , and  $h^0(A_1, \mathcal{O}_{A_1}) = 1$ .

*Proof.* Since  $h^0(A, \mathcal{O}_A) \geq 2$ , there exists a section  $s$  in  $H^0(A, \mathcal{O}_A)$  vanishing identically on some component of  $A$ . Choose a 0-maximal such section  $s$  and let  $A_2 := Z_s$ ,  $A_1 := D - A_2$ . From the exact sequence

$$0 \rightarrow \mathcal{O}_{A_1}(-A_2) \rightarrow \mathcal{O}_A \rightarrow \mathcal{O}_{A_2} \rightarrow 0$$

we get

$$0 \rightarrow H^0(A_1, \mathcal{O}_{A_1}(-A_2)) \rightarrow H^0(A, \mathcal{O}_A) \xrightarrow{r} H^0(A_2, \mathcal{O}_{A_2}).$$

By the hypothesis of 0-maximality of  $s$ , every section of  $H^0(A_1, \mathcal{O}_{A_1}(-A_2))$  does not vanish identically on any component  $\Gamma$  of  $A_1$  and so in particular  $\Gamma A_2 \leq 0$  and  $A_1 A_2 \leq 0$ . This proves assertions ii) and iii).

Furthermore for any  $\Gamma$ , again the hypothesis of 0-maximality implies that the kernel of the restriction map

$$H^0(A_1, \mathcal{O}_{A_1}(-A_2)) \rightarrow H^0(\Gamma, \mathcal{O}_\Gamma(-A_2))$$

is 0-dimensional and therefore we get assertions iv) and v), because

$$h^0(A, \mathcal{O}_A) \leq h^0(A_1, \mathcal{O}_{A_1}(-A_2)) + h^0(A_2, \mathcal{O}_{A_2}).$$

The last assertion is clear, by the previous considerations.  $\square$

**Remarks 1.3.** a) In the decomposition above, if there is a component  $\Gamma$  of  $A_1$  such that  $\Gamma A_2 = 0$ , then  $h^0(A_1, \mathcal{O}_{A_1}(-A_2)) = 1$ . Otherwise again one would get a contradiction to 0-maximality of  $s$ .

b) Lemma 1.2 means that a curve  $D$  such that  $h^0(D, \mathcal{O}_D) \geq 2$  is necessarily not 1-connected. However  $h^0(D, \mathcal{O}_D) = 1$  does not imply 1-connectedness. For instance a multiple fibre  $F = mD$  of a fibration is not 1-connected but  $h^0(F, \mathcal{O}_F) = 1$  (cf. [BPV, Chp. III] )

**Lemma 1.4.** *Let  $D$  be a 1-connected curve. If  $A \prec D$  is such that  $A(D - A) = b$ , then  $h^0(A, \mathcal{O}_A) \leq b$ .*

*Proof.* We do this proof by induction on  $b$ . If  $b = 1$  then, because  $D$  is 1-connected,  $A$  is also and so  $h^0(A, \mathcal{O}_A) = 1$ . We assume that we have proved the assertion for  $m < b$  and we want to prove for  $m = b$ .

Suppose that  $h^0(A, \mathcal{O}_A) \geq 2$ . Take a decomposition of  $A$  as in Lemma 1.2 and let  $A_1 A_2 = -\alpha$ , with  $\alpha \geq 0$ . Note that, because for every component  $\Gamma$  of  $A_1$ ,  $\Gamma A_2 \leq 0$ , we have, for any component  $\Gamma$  of  $A_1$ ,  $\Gamma A_2 \geq -\alpha$ , i.e.  $\Gamma(-A_2) \leq \alpha$ , and so  $h^0(\Gamma, \mathcal{O}_\Gamma(-A_2)) \leq \alpha + 1$ .

Now, by 1-connectedness of  $D$ ,  $A_1(A_2 + (D - A)) \geq 1$ , and so  $A_1(D - A) \geq 1 + \alpha$ . Since  $(A_1 + A_2)(D - A) = b$ ,  $A_2(D - A) \leq b - 1 - \alpha$  and so  $A_2(A_1 + (D - A)) = A_2(D - A_2) \leq b - 1 - 2\alpha$ . Hence by the induction hypothesis  $h^0(A_2, \mathcal{O}_{A_2}) \leq b - 1 - 2\alpha$  and so by Lemma 1.2  $h^0(A, \mathcal{O}_A) \leq b - \alpha$ .  $\square$

**Corollary 1.5.** *Suppose that the curve  $D$  is 1-connected. If  $A \prec D$  is such that  $A(D - A) = b$  and  $h^0(A, \mathcal{O}_A) = b$ , then  $A$  decomposes as  $B_1 + \dots + B_b$ , such that  $\mathcal{O}_{B_i}(-(B_{i+1} + \dots + B_b)) = \mathcal{O}_{B_i}$ , for every  $i = 1, \dots, b-1$ ,  $B_i(D - A) = 1$  and  $B_i$  is 1-connected, for every  $i = 1, \dots, b$ . Furthermore either  $B_i \preceq B_{i+1} + \dots + B_b$  or  $B_i \cap B_{i+1} + \dots + B_b = \emptyset$ .*

*Proof.* Suppose that  $h^0(A, \mathcal{O}_A) = b$ . Then in the decomposition as in the previous proof we must have  $\alpha = 0$  and  $A_1(D - A) = 1$ , meaning that  $A_2(D - A) = b - 1$ . So by Lemma 1.2 necessarily  $h^0(A_1(-A_2)) = 1$  and  $\mathcal{O}_{A_1}(-A_2) = \mathcal{O}_{A_1}$ .

Note also that  $A_1(D - A_1) = 1$  means that  $A_1$  is 1-connected. Assume that  $A_1$  has common components with  $A_2$ . Then we can write  $A_1 = H + B$ ,  $A_2 = H + C$  where  $B$  and  $C$  have no common components and  $H \neq 0$ . Suppose  $B \neq 0$ . Since  $\mathcal{O}_{A_1}(-A_2) = \mathcal{O}_{A_1}$ ,  $B(H + C) = 0$  and

so, because  $BC \geq 0$ , we conclude that  $BH \leq 0$ . But this contradicts the 1-connectedness of  $A_1$  and so  $B = 0$ .

We take  $B_1 := A_1$ . Now we consider  $D - A_1$  which is still 1-connected. One has  $h^0(A_2, \mathcal{O}_{A_2}) = b - 1$  and  $A_2(D - A_1 - A_2) = b - 1$ . We can apply the same reasoning as before and an obvious induction gives us the statement.  $\square$

**Lemma 1.6.** *Let  $\mathcal{L}$  be an invertible sheaf on a curve  $Z$  satisfying  $\deg \mathcal{L}|_{Z'} \geq 2p_a(Z') - 2$  for any  $0 \prec Z' \preceq Z$ . If  $h^1(Z, \mathcal{L}) \neq 0$ , then there exists a subcurve  $A \preceq Z$  with  $\mathcal{L}|_A \simeq \omega_A$  and  $h^0(A, \mathcal{O}_A) = 1$ .*

*Proof.* By duality, we have  $h^0(Z, \omega_Z \otimes \mathcal{L}^{-1}) \neq 0$ . Take a 0-maximal  $s \in H^0(Z, \omega_Z \otimes \mathcal{L}^{-1})$  and put  $A = Z - Z_s$  (possibly  $Z_s = 0$ ). Then  $\mathcal{O}_A(-Z_s) \otimes \omega_Z \otimes \mathcal{L}^{-1} \simeq \omega_A \otimes \mathcal{L}^{-1}$  and  $s$  induces a non-zero  $s' \in H^0(A, \omega_A \otimes \mathcal{L}^{-1})$  which does not vanish identically on any component of  $A$ . In particular,  $\omega_A \otimes \mathcal{L}^{-1}$  is nef. Since  $\deg \omega_A = 2p_a(A) - 2 \leq \deg \mathcal{L}|_A$  by the assumption,  $\omega_A \otimes \mathcal{L}^{-1}$  is numerically trivial. Furthermore, since  $s'$  is nowhere vanishing, we get  $\omega_A \otimes \mathcal{L}^{-1} \simeq \mathcal{O}_A$ . By the 0-maximality of  $s$ , the restriction map  $H^0(A, \omega_A \otimes \mathcal{L}^{-1}) = H^0(A, \mathcal{O}_A) \rightarrow H^0(\Gamma, \mathcal{O}_\Gamma)$  is injective for any irreducible component  $\Gamma \preceq A$ . Hence  $h^0(A, \mathcal{O}_A) = 1$ .  $\square$

**Proposition 1.7.** *Let  $Z$  be a curve such that for any  $0 \prec Z' \preceq Z$ ,  $p_a(Z') \leq 0$ . Then:*

- i) *every component  $\Gamma$  of  $Z$  is a smooth rational curve;*
- ii) *for any  $0 \prec Z' \preceq Z$ ,  $h^1(Z', \mathcal{O}_{Z'}) = 0$ .*

*Furthermore,  $p_a(Z) = 0$  (and  $h^0(Z, \mathcal{O}_Z) = 1$ ) if and only if  $Z$  is 1-connected.*

*Proof.* By Lemma 1.6 applied to  $\mathcal{L} = \mathcal{O}_Z$ , if  $h^1(Z, \mathcal{O}_Z) \neq 0$  there is a subcurve  $A$  with  $\mathcal{O}_A \simeq \omega_A$  and  $h^0(A, \mathcal{O}_A) = 1$ . This implies  $p_a(A) = 1$  contradicting the hypothesis  $p_a(A) \leq 0$ . Therefore,  $h^1(Z, \mathcal{O}_Z) = 0$ . Then we get i) and ii), since the natural map  $H^1(Z, \mathcal{O}_Z) \rightarrow H^1(Z', \mathcal{O}_{Z'})$  is surjective for any  $0 \prec Z' \preceq Z$ .

If  $Z$  is 1-connected, then  $h^0(Z, \mathcal{O}_Z) = 1$  (cf. Lemma 1.2) and we have  $p_a(Z) = 0$  by  $h^1(Z, \mathcal{O}_Z) = 0$ . Conversely, assume that  $p_a(Z) = 0$ . Then  $0 = p_a(Z) = p_a(Z_1) + p_a(Z_2) - 1 + Z_1 Z_2 \leq -1 + Z_1 Z_2$  for any decomposition  $Z = Z_1 + Z_2$  with  $0 \prec Z_1, Z_2$ . Hence  $Z$  is 1-connected.  $\square$

2. FIXED COMPONENTS OF  $|\omega_D|$ 

The results in this section prove Theorem 0.1.

**Proposition 2.1.** *Let  $D$  be a 1-connected curve and  $Z \prec D$  a curve such that the restriction map  $H^0(D, \omega_D) \rightarrow H^0(Z, \omega_D)$  is the zero map. Then  $p_a(Z) \leq 0$  and  $h^0(D - Z, \mathcal{O}_{D-Z}) = (D - Z)Z + p_a(Z)$ .*

*Proof.* Let  $B := D - Z$ . As usual one has  $p_a(D) = p_a(B) + p_a(Z) - 1 + BZ$ , and so  $p_a(B) - 1 = p_a(D) - BZ - p_a(Z)$ .

Since the kernel of the restriction map  $H^0(D, \omega_D) \rightarrow H^0(Z, \omega_D)$  is exactly  $H^0(B, \omega_B)$ , our hypothesis implies that  $h^0(B, \omega_B) = h^0(D, \omega_D)$ . The equality  $p_a(B) - 1 = h^0(B, \omega_B) - h^0(B, \mathcal{O}_B)$  yields then

$$p_a(D) - h^0(B, \mathcal{O}_B) = p_a(D) - BZ - p_a(Z),$$

i.e.,  $h^0(B, \mathcal{O}_B) = BZ + p_a(Z)$ .

By Lemma 1.4,  $h^0(B, \mathcal{O}_B) \leq BZ$  and so  $p_a(Z) \leq 0$ .  $\square$

**Corollary 2.2.** *Let  $D$  be a 1-connected curve and let  $Z$  be the fixed part of  $|\omega_D|$ . Then for any  $Z' \preceq Z$ ,  $p_a(Z') \leq 0$  and  $h^1(Z', \mathcal{O}_{Z'}) = 0$ .*

*Proof.* The statement is an immediate consequence of Propositions 2.1 and 1.7.  $\square$

**Corollary 2.3.** *Let  $D$  be a 1-connected curve and  $Z \prec D$  a 1-connected curve (or more generally such that  $h^0(Z, \mathcal{O}_Z) = 1$ ) such that the restriction map  $H^0(D, \omega_D) \rightarrow H^0(Z, \omega_D)$  is the zero map. Then  $h^1(Z, \mathcal{O}_Z) = 0$  and  $B := D - Z$  decomposes as in Corollary 1.5.*

*Proof.* By Corollary 2.2,  $p_a(Z) \leq 0$  and  $h^1(Z, \mathcal{O}_Z) = 0$ , and so, because  $h^0(Z, \mathcal{O}_Z) = 1$ , we have  $p_a(Z) = 0$ . By Proposition 2.1,  $h^0(B, \mathcal{O}_B) = BZ$  and therefore we can apply Corollary 1.5 obtaining a decomposition as wished.  $\square$

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